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Inertial range scaling in turbulence

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In this paper, we apply She and Leveque's [Z.-S. She and E. Leveque, Phys. Rev. Lett. **72**, 336 (1994)] hierarchy model under the assumption that $\lim_{p\to\infty}\tau_p/p=-1$ with τ_p being the scaling exponent for the local averaged dissipation function suggested by Novikov [E. A. Novikov, Phys. Rev. E **50**, R3303 (1994)]. The resulting model agrees well with existing theoretical and experimental results for $p\leqslant 10$. The most interesting prediction of this model is the saturation of the exponents of the velocity increment as $p\to\infty$.

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In fully developed turbulence flows, it is believed that there exists a power law scaling range for the velocity increment in the inertial range [1], i.e.,

$$\langle \delta u_r^p\rangle=\mathrm{const}\times r^{\xi_p}, \tag{1}$$

where $\langle\,\rangle$ denotes an ensemble average, r is the separation, and $\delta u_r=u(x+r)-u(x)$. According to Kolmogorov [2], the velocity increment is related to the local averaged dissipation rate, ϵ_r, which scales as

$$\langle \epsilon_r^p\rangle=\mathrm{const}\times r^{\tau_p}, \tag{2}$$

and the relationship between these two exponents can be written as [3]

$$\xi_p=p/3+\tau_{p/3}. \tag{3}$$

She and Leveque [4] (SL) have recently proposed a hierarchy structure model yielding the scaling exponents for the velocity increment,

$$\xi_p=\frac{p}{9}+2\left[1-\left(\frac{2}{3}\right)^{p/3}\right], \tag{4}$$

in good agreement with experimental values up to measurable orders ($p\leqslant 10$). On the other hand, Novikov [5] applied the theory of infinitely divisible distribution to the scaling properties of ϵ_r. He suggested that the parameter

$$h=-\lim_{p\to\infty}\tau_p/p \tag{5}$$

should equal 1. He claims that $h=2/3$ in the SL model implies a gap in the probability distribution function (PDF) of the multiplicative factor, ϵ_r/ϵ_l, which contradicts to existing experiments [6]. Using the model by Novikov, Nelkin [7] calculated scaling exponents, ξ_p, which are indistinguishable for $p\leqslant 10$ from the exponents by She and Leveque. Nelkin argues that the underlying physics of the most intensive events in these models are quite different.

In this paper, we generalize the SL model by letting h be a free parameter. We prove that if the hierarchy structure model is correct, then $h=1$ implies a saturation of scaling exponents for $p\to\infty$. On the other hand, if $h=1$ is a rigorous mathematical theorem, then the hierarchy structure should yield a saturation of the scaling exponents.

First, we derive the generalized SL model following [4]. Assuming that the hierarchy of ϵ_r^p in the original SL model is still valid, we have

$$\epsilon_r^{(p+1)}=A_p\epsilon_r^{(p)^{\beta}}\epsilon_r^{(\infty)^{1-\beta}},\quad 0<\beta<1 \tag{6}$$

where $\epsilon_r^{(p)}=\langle\epsilon_r^{p+1}\rangle/\langle\epsilon_r^p\rangle$, β is a constant, and A_p are functions only depending on p. Using (2), we obtain the difference equation for τ_p,

$$\tau_{p+2}-(1+\beta)\tau_{p+1}+\beta\tau_p+q(1-\beta)=0. \tag{7}$$

Here the constant q describes the scaling exponent for the most intensive structures through the following relation:

$$\epsilon_r^{(\infty)}\sim r^{-q}. \tag{8}$$

We assume that q is a free parameter and is to be determined later. An apparent family of solutions of (7) can be written as

$$\tau_p=ap+b+f(p), \tag{9}$$

where a and b are constants, and $f(p)$ is a polynomial function of p. They are to be determined. Clearly, the first two terms in (9) are special solutions for the inhomogeneous difference equation (7) when $a=-q$, and $f(p)$ satisfies a homogeneous equation, i.e.,

$$f(p+2)-(1+\beta)f(p+1)+\beta f(p)=0, \tag{10}$$

yielding a nontrivial solution $f(p)=\alpha\beta^p$, where α is a constant. If τ_p grows more slowly than exponential, we have $f(\infty)=0$ and $\beta<1$. The general solution of (7) has, then, the form

$$\tau_p=-qp+b+\alpha\beta^p, \tag{11}$$

subject to boundary conditions $\tau_0=0$ and $\tau_1=0$. The first condition is trivial and the second condition is from Kolmogorov [2] and the 4/5 law [1]. If we further use the definition of intermittency parameter, $\mu=-\tau_2$ and $h=-\lim_{p\to\infty}\tau_p/p$

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defined by Novikov [5], we have $q=h$, $b=-\alpha=h^2/\mu$, and $\beta=1-\mu/h$. Equation (11) can be rewritten as

$$\tau_p=-hp+\frac{h^2}{\mu}\left[1-\left(1-\frac{\mu}{h}\right)^p\right]. \quad (12)$$

It should be pointed out that the model derived here is realizable in the sense of the non-negative PDF for the multiplicative factor. In fact, (12) corresponds to Eq. (14) in Ref. [5] if the density function is chosen as

$$F'(x)=A\,\delta(x-x_0), \quad (13)$$

where $\delta(x)$ is the Dirac delta function, A and x_0 are constants, depending on h and μ:

$$x_0=-\ln\left(1-\frac{\mu}{h}\right),$$

and

$$A=-\frac{h^2}{\mu}\ln\left(1-\frac{\mu}{h}\right).$$

Therefore, the realizability of the model can be guaranteed by the Lévy-Baxter-Shapiro theorem [5]. Using the δ function as the Lévy-Khinchine measure has been proposed in the model by She and Waymire [8], leading to a log-Poisson distribution for the multiplicative factor.

Using Ref. [2], the scaling exponents for the velocity increment can be written as

$$\xi_p=\frac{1}{3}(1-h)p+\frac{h^2}{\mu}\left[1-\left(1-\frac{\mu}{h}\right)^{p/3}\right]. \quad (14)$$

Up to now, the only assumption used is the hierarchy relationship (6). $h(\geq\mu/2)$ and $\mu<1$ are two positive free parameters. The original SL model corresponds to $h=2/3$ and $\mu=2/9$. If we follow Novikov's suggestion to choose $h=1$, (12) and (14) become

$$\tau_p=-p+\frac{1}{\mu}[1-(1-\mu)^p], \quad (15)$$

and

$$\xi_p=\frac{1}{\mu}[1-(1-\mu)^{p/3}], \quad (16)$$

An interesting consequence of (16) is that ξ_p saturates to $1/\mu$ when $p\to\infty$.

Specifically, if we choose $\mu=2/9$ which has been confirmed by many experiments [9] and used in the models by She-Leveque and Nelkin, then $\beta=7/9$ and we get a new formula for the scaling exponents:

$$\xi_p=\frac{9}{2}\left[1-\left(\frac{7}{9}\right)^{p/3}\right]. \quad (17)$$

The prediction of this model is consistent with the original SL model and Nelkin's model [7] for $p\leq10$, as listed in Table I. For $p>20$, the scaling exponents begin to approach the saturation limit (see Fig. 1).

TABLE I. Scaling exponents ξ_p for various models.

Order p	Eq. (17)	SL	Kolmogorov [2]	Nelkin [7]
1	0.361	0.363	0.358	0.373
2	0.694	0.695	0.691	0.702
3	1.000	1.000	1.000	1.000
4	1.281	1.279	1.283	1.273
5	1.539	1.538	1.543	1.528
6	1.777	1.777	1.777	1.767
7	1.996	2.001	1.987	1.993
8	2.197	2.210	2.172	2.208
9	2.382	2.407	2.333	2.414
10	2.552	2.593	2.469	2.611

It is noticed that for $\mu=2/9$ the values of the low-order scaling exponents, ξ_p $(p\leq10)$, in (14) are not sensitive to h when h varies from 2/3 to 1. As a matter of the fact, the variation is within 2%. This observation implies that h is a measure for the most intensive structures which contributes primarily to high-order statistics. In addition, from (3) and the assumption $h=1$, we can obtain $\lim_{p\to\infty}\zeta_p/p=0$, indicating that the growth of ζ_p must be slower than a linear function for p large enough. From (12) and (14), we obtain $\tau_p(\mu\to0)=0$ and $\xi_p(\mu\to0)=p/3$ regardless of h, indicating that the hierarchy structure model is a natural extension of the theory of Kolmogorov [1].

A cutoff of the scaling exponents of velocity increment in the one-dimensional Burgers equation has been observed [10]. Its physical origin in connection with the shock structure has been studied. At this point, we are not sure what the underlying physics is for the saturation of the scaling exponents in the three-dimensional incompressible Navier-Stokes turbulence. Nevertheless, the saturation of the scaling exponents might indicate a similar physics: the direct connection of the characteristic length of the structure with its intensity for the most intensive events.

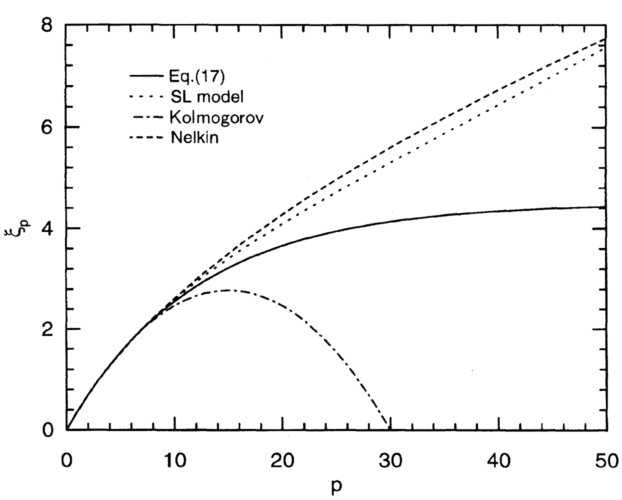

FIG. 1. Scaling exponents of velocity increment, ξ_p, as a function of p for Eq. (17), SL model, Kolmogorov [2] and Nelkin's formula [7].

The fundamental assumption of the hierarchy structure model in (7) requires that the scaling exponents approach their asymptotic values with an exponential rate, which differs from the model by Nelkin and the passive-scalar turbulence [11]. The rate of approach to the asymptotic values is determined by the tails of the corresponding PDFs. Since the events in that range are extremely rare, it seems difficult to resolve this issue soon through experimental or numerical studies.

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